ECS315 2019/1   Part III.2   Dr.Prapun

popular and useful

8.3   Families of Discrete Random Variables

Many physical systems can be modeled by the same or similar random experiments and random variables. In this subsection, we present the analysis of several discrete random variables that frequently arise in applications.[34]

uniform
Bernoulli
binomial
geometric
Poisson

Definition 8.26. X is ***uniformly distributed*** on a finite set S if

$$p_X(x) = P[X = x] = \begin{cases} \frac{1}{|S|} & x \in S, \\ 0, & \text{otherwise,} \end{cases}$$

equally likely   *classical probability*

- We write $X \sim \mathcal{U}(S)$ or $X \sim \text{Uniform}(S)$.

  the symbol "~" can be read as "is drawn from" or "is distributed following"

- Read "X is uniform on S" or
  "X is a uniform random variable on set S".

- The pmf is usually referred to as the uniform discrete distribution.

- Simulation: When the support S contains only consecutive integers[35], it can be generated by the command `randi` in MATLAB (R2008b).

---

[34]As mention in 7.22, we often omit a discussion of the underlying sample space of the random experiment and directly describe the distribution of a particular random variable.

[35]or, with minor manipulation, only uniformly spaced numbers

Example 8.27. X is uniformly distributed on $1, 2, \ldots, n$ if

$$p_X(x) = \begin{cases} \frac{1}{n}, & x = 1, 2, 3 \ldots, n \\ 0, & \text{otherwise.} \end{cases}$$

In MATLAB, X can be generated by `randi(n)`.

Example 8.28. Uniform pmf is used when the random variable can take finite number of "equally likely" or "totally random" values.

- Classical game of chance / classical probability

- Fair gaming devices (well-balanced coins and dice, well-shuffled decks of cards)

Example 8.29. Roll a fair dice. Let X be the outcome. $X \sim \mathcal{U}(\{1, 2, \ldots, 6\})$

$$p_X(x) = \begin{cases} \frac{1}{6}, & x = 1, 2, \ldots, 6 \\ 0, & \text{otherwise.} \end{cases}$$
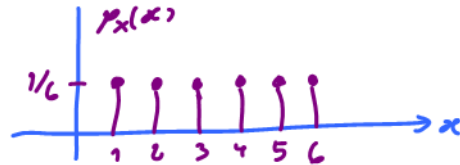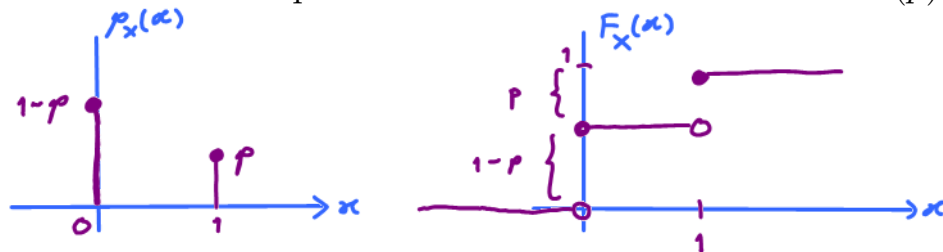

Definition 8.30. X is a ***Bernoulli*** random variable if

$$p_X(x) = \begin{cases} 1 - p, & x = 0, \\ p, & x = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad p \in (0, 1)$$

- Write $X \sim \mathcal{B}(1, p)$ or $X \sim \text{Bernoulli}(p)$

- X takes only two values: 0 or 1

Example 8.31. Plot the pmf and cdf of the RV $X \sim \text{Bernoulli}(p)$.

Definition 8.32. X is a ***binary*** random variable if

$$p_X(x) = \begin{cases} 1 - p, & x = a, \\ p, & x = b, \\ 0, & \text{otherwise,} \end{cases} \qquad p \in (0, 1), \quad b > a.$$

113

- X takes only two values: a or b

Definition 8.33. X is a **_binomial_** random variable with parameters $n \in \mathbb{N}$ and $p \in (0, 1)$ if

$$p_X(x) = \begin{cases} \binom{n}{x}p^x(1-p)^{n-x}, & x \in \{0, 1, 2, \ldots, n\}, \\ 0, & \text{otherwise} \end{cases} \qquad (14)$$

check ① $p_x(x) \geqslant 0$ ✓

② "$\sum = 1$" $\quad \sum_x p_x(x) = \sum_{x=0}^{n} \binom{n}{x} \underbrace{p}_{a}{}^x \underbrace{(1-p)}_{b}{}^{n-x} = (p+(1-p))^n = 1^n = 1$

- Write $X \sim \mathcal{B}(n, p)$ or $X \sim \text{binomial}(n, p)$.

$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$ binomial theorem

  - Observe that $\mathcal{B}(1, p)$ is Bernoulli with parameter p.

- To calculate $p_X(x)$, can use `binopdf(x,n,p)` in MATLAB.

- Interpretation: X is the number of successes in n independent Bernoulli trials. The formula follows directly from 6.55.

- Simulation: To generate such RV, use `binornd(n,p)` in MATLAB.

Example 8.34. $X \sim \text{binomial}(10, 0.3)$

$X \sim \text{binomial}(10, 0.3) \iff p_X(x) = \binom{10}{x} 0.3^x (1-0.3)^{10-x}$
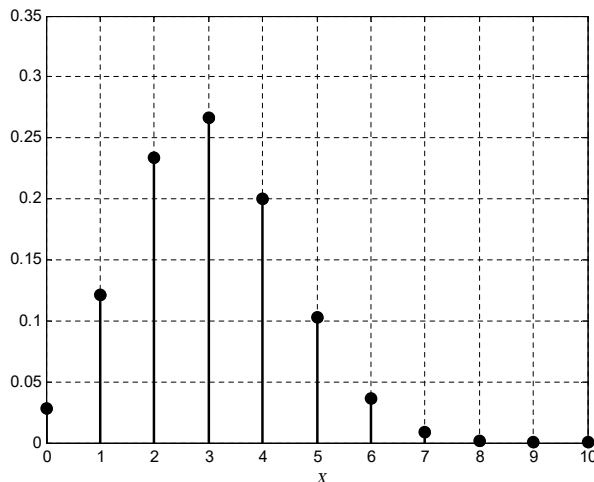
Figure 21: The expression and the plot of the pmf for $X \sim$ binomial $(10, 0.3)$


```
p = binopdf(x,10,0.3)
stem(x,p,'k','filled','LineWidth',1.5); grid on
```

114

Example 8.35. Find $P[X > 3]$ and $P[X > 0]$ for the following random variables:

(a) $X \sim \text{Uniform}(\{1, 2, \ldots, 15\}) \Rightarrow p_X(x) = \begin{cases} \frac{1}{15}, & x \in \{1, 2, \ldots, 15\}, \\ 0, & \text{otherwise.} \end{cases}$

$P[X > 3] = \sum\limits_{x=4}^{15} p_X(x) = \frac{1}{15} \times 12 = \frac{12}{15} = \frac{4}{5} = 0.8$

$P[X > 0] = 1 \leftarrow$ Every possible values of X are > 0.

(b) $X \sim \text{Bernoulli}(0.7) \Rightarrow p_X(x) = \begin{cases} 0.7, & x = 1, \\ 0.3, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$

$P[X > 3] = 0 \leftarrow X$ can only be 0 or 1. Both values are < 3.

$P[X > 0] = p_X(1) = 0.7$

(c) $X \sim \text{binomial}\left(2, \frac{1}{2}\right) \Rightarrow p_X(x) = \begin{cases} \binom{2}{x} p^x (1-p)^{2-x} = \binom{2}{x}\left(\frac{1}{2}\right)^2, & x = 0, 1, 2, \\ 0, & \text{otherwise.} \end{cases}$

$P[X > 3] = 0$

$P[X > 0] = p_X(1) + p_X(2) = 1 - p_X(0) = 1 - \binom{2}{0}\left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$

Example 8.36. An optical inspection system is to distinguish among different part types. The probability of a correct classification of any part is 0.98. Suppose that three parts are inspected and that the classifications are independent.

(a) Let the random variable X denote the number of parts that are correctly classified. Determine the probability mass function of X. [15, Q3-20]

(b) Let the random variable Y denote the number of parts that are incorrectly classified. Determine the probability mass function of Y.

Solution:

(a) X is a binomial random variable with $n = 3$ and $p = 0.98$. Hence,

$$p_X(x) = \begin{cases} \binom{3}{x} 0.98^x (0.02)^{3-x}, & x \in \{0, 1, 2, 3\}, \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

In particular, $p_X(0) = 8 \times 10^{-6}$, $p_X(1) = 0.001176$, $p_X(2) = 0.057624$, and $p_X(3) = 0.941192$. Note that in `MATLAB`, these probabilities can be calculated by evaluating `binopdf(0:3,3,0.98)`.

(b) Y is a binomial random variable with $n = 3$ and $p = 0.02$. Hence,

$$p_Y(y) = \begin{cases} \binom{3}{y} 0.02^y (0.98)^{3-y}, & y \in \{0, 1, 2, 3\}, \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

In particular, $p_Y(0) = 0.941192$, $p_Y(1) = 0.057624$, $p_Y(2) = 0.001176$, and $p_Y(3) = 8 \times 10^{-6}$. Note that in `MATLAB`, these probabilities can be calculated by evaluating `binopdf(0:3,3,0.02)`.

Alternatively, note that there are three parts. If X of them are classified correctly, then the number of incorrectly classified parts is $n - X$, which is what we defined as Y. Therefore, $Y = 3 - X$. Hence, $p_Y(y) = P[Y = y] = P[3 - X = y] = P[X = 3 - y] = p_X(3 - y)$.

Example 8.37. Daily Airlines flies from Amsterdam to London every day. The price of a ticket for this extremely popular flight route is $75. The aircraft has a passenger capacity of 150. The airline management has made it a policy to sell 160 tickets for this flight in order to protect themselves against no-show passengers. Experience has shown that the probability of a passenger being a no-show is equal to 0.1. The booked passengers act independently of each other. Given this overbooking strategy, what is the probability that some passengers will have to be bumped from the flight?

Solution: This problem can be treated as 160 independent trials of a Bernoulli experiment with a success rate of $p = 9/10$, where a passenger who shows up for the flight is counted as a success. Use the random variable X to denote number of passengers that show up for a given flight. The random variable X is binomial distributed with the parameters $n = 160$ and $p = 9/10$. The probability in question is given by

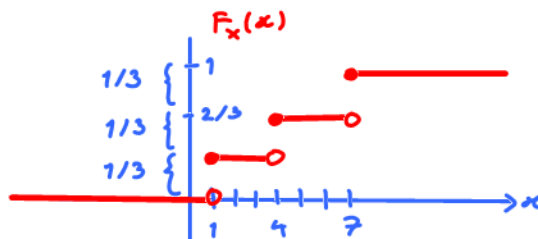$$P[X > 150] = 1 - P[X \leq 150] = 1 - F_X(150).$$

In `MATLAB`, we can enter `1-binocdf(150,160,9/10)` to get 0.0359. Thus, the probability that some passengers will be bumped from any given flight is roughly 3.6%. [21, Ex 4.1]

X is a uniform RV

Example 8.38. Suppose $X \sim \mathcal{U}(\{1, 4, 7\})$. Plot the cdf of X.

This is its support S_X

$$P_X(x) = \begin{cases} 1/3, & x = 1, 4, 7, \\ 0, & \text{otherwise.} \end{cases}$$

$F_X(x)$

Example 8.39. Suppose $X \sim$ binomial $(2, 0.2)$

(a) What are the possible values of X?

$$S_X = \{0, 1, 2, \ldots, n\} = \{0, 1, 2\}$$

(b) Find the (default) support of X.

$$\{0, 1, 2\}$$

(c) Find $P[X > 2] = 0$

none of the possible values of X is greater than 2.

Example 8.40 (M2013Q6)**.** Consider a sample space $\Omega = \{1, 2, 3\}$.
Suppose, for $\omega = 1, 2, 3$, we have

CH5

$$P(\{\omega\}) = \frac{c - |\omega - 2|}{4}$$

for some constant c.

CH7   Define ~~another~~ random variable Y by $Y(\omega) = \sqrt{|\omega - 2|}$.

ω	$P(\{\omega\})$
1	$\frac{c-1}{4} = \frac{1}{4}$
2	$\frac{c}{4} = \frac{1}{2}$
3	$\frac{c-1}{4} = \frac{1}{4}$

$P(\Omega) = 1$

$\frac{c-1}{4} + \frac{c}{4} + \frac{c-1}{4} = 1$

$3c - 2 = 4$

$c = 2$

(a) Is Y a uniform random variable?

Yes because all of its possible values are equally likely.

(b) Is Y a Bernoulli random variable?

Yes because it can be 0 or 1 only.

CH8

(c) Is Y a Binomial random variable?

Yes because it is a Bernoulli RV.

Any Bernoulli RV is a binomial RV.

$(n = 1)$

117

ω	$Y(\omega) = \sqrt{|\omega - 2|}$
1	1
2	0
3	1

$p_Y(0) \equiv P[Y = 0] = P(\{2\}) = 1/2$

$p_Y(1) \equiv P[Y = 1] = P(\{1, 3\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$p_Y(y) = \begin{cases} 1/2, & y = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

0010----..

3

2

8.41. Defining discrete random variables from the Bernoulli trials:

n = 12

Here, X = 5

01000101100010111⟨000101011100⟩...

Here, X = 4

G₁ The number of <u>trials</u> until the **next 1** is a geometric$_1$ random variable.

The number of 1s in n trials is a binomial random variable with parameter (n,p)

In the limit, as $n \to \infty$ and $p \to 0$ while $np = \alpha$,

G₀ The number of <u>0</u> until the **next 1** is a geometric$_0$ random variable.

Here, X = 3

The number of 1s is a Poisson random variable with parameter $\alpha = np$.

Figure 22: Many discrete random variables discussed in our class can be defined via the Bernoulli trials. Here, successes and failures are denoted by 1 and 0, respectively.

Definition 8.42. A geometric random variable X is defined by the fact that for some constant $\beta \in (0, 1)$,

$$p_X(k + 1) = \beta \times p_X(k)$$

for all $k \in S$ where S can be either $\mathbb{N}$ or $\mathbb{N} \cup \{0\}$.

(a) When its support is $\mathbb{N} = \{1, 2, \ldots\}$,

$$p_X(x) = \begin{cases} (1 - \beta)\,\beta^{x-1}, & x = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

- In MATLAB, to evaluate $p_X(x)$, use geopdf(x-1,1-β).
- Interpretation: <mark>X is the number of trials required in Bernoulli trials to achieve the first success.</mark>

S₁

"X = k" is equivalent to

{ the first k-1 trials : failure

118

and

the kᵗʰ trial : success

In particular, in a series of Bernoulli trials (independent trials with constant probability p of a success), let the random variable X denote the number of trials until the first success. Then X is a geometric random variable with $\beta = 1 - p$ and

$$p_X(x) = \begin{cases} (1-\beta)\beta^{x-1}, & x = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} p(1-p)^{x-1}, & x = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

- Write $X \sim \mathcal{G}_1(p)$ or geometric$_1(p)$.

(b) When its support is $\mathbb{N} \cup \{0\}$,

$$p_X(x) = \begin{cases} (1-\beta)\beta^{x}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} p(1-p)^{x}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

- Write $X \sim \mathcal{G}_0(p)$ or geometric$_0(p)$.
- In MATLAB, to evaluate $p_X(x)$, use geopdf(x,1-β).
- Interpretation: X is the number of failures in Bernoulli trials before the first success occurs.

Handwritten annotations (left margin): $\mathcal{G}_1$, $\mathcal{G}_0$

Handwritten annotations (right margin):

check:

① $p_X(x) \geq 0$ ✓

②
$$\sum_{x=1}^{\infty} p(1-p)^{x-1}$$
$$m = x - 1$$
$$= \sum_{m=0}^{\infty} p(1-p)^{m}$$
$$= p \sum_{m=0}^{\infty} (1-p)^{m} = p \frac{1}{1-(1-p)}$$
$$= 1 \checkmark$$

$$\sum_{m=0}^{\infty} r^{m} = \frac{1}{1-r}$$

$A = 1 + r + r^2 + \cdots$
$rA = r + r^2 + \cdots$
$(1-r)A = 1$ (assuming $|r| < 1$)
$A = \frac{1}{1-r}$

8.43. In 1837, the famous French mathematician Poisson introduced a probability distribution that would later come to be known as the Poisson distribution, and this would develop into one of the most important distributions in probability theory. As is often remarked, Poisson did not recognize the huge practical importance of the distribution that would later be named after him. In his book, he dedicates just one page to this distribution. It was Bortkiewicz in 1898, who first discerned and explained the importance of the Poisson distribution in his book Das Gesetz der Kleinen Zahlen (*The Law of Small Numbers*). [21]

Definition 8.44. X is a **Poisson** random variable with **parameter** $\alpha > 0$ if

$$p_X(x) = \begin{cases} e^{-\alpha}\frac{\alpha^x}{x!}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

check:
① $p_X(x) \geq 0$ ✓
② "$\sum = 1$"?

$$\sum_{x=0}^{\infty} e^{-\alpha}\frac{\alpha^x}{x!} = e^{-\alpha}\underbrace{\sum_{x=0}^{\infty}\frac{\alpha^x}{x!}}_{e^{\alpha}} = 1 \checkmark$$

- In MATLAB, use `poisspdf(x,alpha)`.

- Write $X \sim \mathcal{P}(\alpha)$ or Poisson(α).

- We will see later in Example 9.7 that α is the "average" or expected value of X. Recall that, from calculus, $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

- Instead of X, Poisson random variable is usually denoted by Λ. The parameter α is often replaced by $\lambda\tau$ where λ is referred to as the **intensity/rate parameter** of the distribution

Example 8.45. The first use of the Poisson model is said to have been by a Prussian (German) physician, Bortkiewicz, who found that the annual number of late-19th-century Prussian (German) soldiers kicked to death by horses fitted a Poisson distribution [6, p 150],[3, Ex 2.23][36].

Example 8.46. The number of hits to a popular website during a 1-minute interval is given by $N \sim \mathcal{P}(\alpha)$ where $\alpha = 2$.

(a) Find the probability that there is at least one hit between 3:00AM and 3:01AM.

$$P[N \geq 1] = p_N(1) + p_N(2) + p_N(3) + \cdots = 1 - P[N < 1]$$
$$= 1 - p_N(0) = 1 - e^{-\alpha}\frac{\alpha^0}{0!} = 1 - e^{-\alpha} = 1 - e^{-2} \approx 0.865$$

(b) Find the probability that there are at least 2 hits during the time interval above.

$$P[N \geq 2] = 1 - P[N < 2] = 1 - \left(p_N(0) + p_N(1)\right) = 1 - \left(e^{-\alpha}\frac{\alpha^0}{0!} + e^{-\alpha}\frac{\alpha^1}{1!}\right)$$
$$= 1 - e^{-\alpha}(1 + \alpha) = 1 - 3e^{-2} \approx 0.594$$

---

[36] I. J. Good and others have argued that the Poisson distribution should be called the Bortkiewicz distribution, but then it would be very difficult to say or write.

8.47. One of the reasons why Poisson distribution is important is because many natural phenomenons can be modeled by **_Poisson processes_**.

8.48. A **_Poisson process_** (PP) is a particular kind of random arrangement of "marks" (denoted by "×" or "⋆") on the time line.
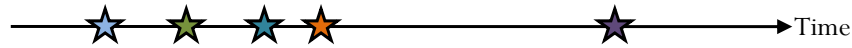
The "marks" may indicate the arrival times or occurrences of event/phenomenon of interest.

Example 8.49. Examples of processes that can be modeled by **_Poisson process_** include

(a) the sequence of times at which lightning strikes occur or mail carriers get bitten within some region

(b) the emission of particles from a radioactive source

(c) the arrival of

- telephone calls at a switchboard or at an automatic phone-switching system
- urgent calls to an emergency center
- (filed) claims at an insurance company
- incoming spikes (action potential) to a neuron in human brain

(d) the occurrence of

- serious earthquakes
- traffic accidents
- power outages

in a certain area.

(e) page view requests to a website

8.50. It is convenient to consider the Poisson process in terms of customers arriving at a facility.

We focus on a type of Poisson process that is called *homogeneous Poisson process.*

Definition 8.51. For *homogeneous Poisson process*, there is only one parameter that describes the whole process. This number is call the *rate* and usually denoted by λ.

Example 8.52. If you think about modeling customer arrival as a Poisson process with rate $\lambda = 5$ customers/hour, then it means that during any fixed time interval of duration 1 hour (say, from noon to 1PM), you expect to have about 5 customers arriving in that interval. If you consider a time interval of duration two hours (say, from 1PM to 3PM), you expect to have about $2 \times 5 = 10$ customers arriving in that time interval.

8.53. One important fact which we will revisit later is that, for a homogeneous Poisson process, the number of arrivals N during a time interval of duration T is a Poisson random variable with parameter $\alpha = \lambda T$.
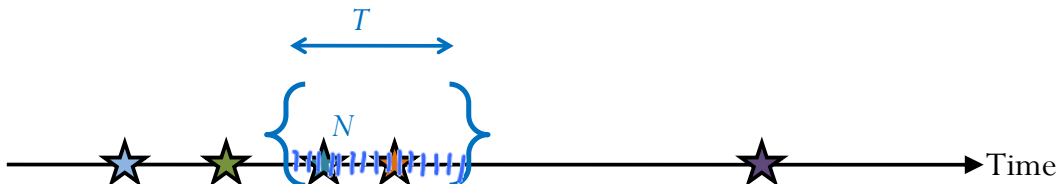
Example 8.54. Examples of Poisson *random variables*:

- #photons emitted by a light source of intensity λ [photon-s/second] in time τ

- #atoms of radioactive material undergoing decay in time τ

- #clicks in a Geiger counter in τ seconds when the average number of click in 1 second is λ.

- #dopant atoms deposited to make a small device such as an FET

- #customers arriving in a queue or workstations requesting service from a file server in time τ

- Counts of demands for telephone connections in time τ

- Counts of defects in a semiconductor chip.

Example 8.55. Thongchai produces a new hit song every 7 months on average. Assume that songs are produced according to a Poisson process. Find the probability that Thongchai produces more than two hit songs in 1 year.

$N = $ # hit songs in 1 year.

$N \sim P(\alpha)$ where $\alpha = \lambda T$

$\lambda = \dfrac{1}{7 \text{ months}}$

$T = 1 \text{ year}$

$= \dfrac{1}{7 \text{ months}} \times 1 \text{ yr}$

$P[N > 2] = 1 - P[N \leq 2]$

$= 1 - \left(e^{-\alpha} \dfrac{\alpha^0}{0!} + e^{-\alpha} \dfrac{\alpha^1}{1!} + e^{-\alpha} \dfrac{\alpha^2}{2!} \right)$

$= \dfrac{12}{7}$

$\alpha = \dfrac{12}{7}$

≈ 0.2466

8.56. ==*Poisson approximation* of Binomial distribution==: When ==p is small and n is large, $B(n,p)$ can be approximated by $P(np)$==

$\alpha = np$

(a) In a large number of independent repetitions of a Bernoulli trial having a small probability of success, the total number of successes is approximately Poisson distributed with parameter $\alpha = np$, where n = the number of trials and p = the probability of success. [21, p 109]

(b) More specifically, suppose $X \sim B(n, p_n)$. If $p_n \to 0$ and $np_n \to \alpha$ as $n \to \infty$, then for $x = 0, 1, 2, \ldots$, we have

$$P[X = x] = \binom{n}{x} p_n^x (1 - p_n)^{n-x} \xrightarrow{n \to \infty} e^{-\alpha} \frac{\alpha^x}{x!}.$$

To see this, note that the first x (largest) terms of $n!$ is of the form $x - k$ where $k = 0, 1, 2, \ldots, x - 1$. Each of them can be bounded by $n - x \leq n - k \leq n$.

123

Therefore, $\frac{(n-x)^x}{x!} \leq \binom{n}{x} \leq \frac{n^x}{x!}$. For example, $\frac{4^3}{3!} \leq \binom{7}{3} = \frac{7 \times 6 \times 5}{3!} \leq \frac{7^3}{3!}$. Hence,

$$P[X = x] = \underbrace{\binom{n}{x}\frac{1}{n^x}}_{\to \frac{1}{x!}} \underbrace{(np_n)^x}_{\to \alpha^x} \underbrace{(1-p_n)^n}_{=(1-\frac{np_n}{n})^n \to e^{-\alpha}} \underbrace{(1-p_n)^{-x}}_{\to 1}.$$

Example 8.57. Consider $X \sim \mathcal{B}(n, 1/n)$. (We have already seen this in Example 6.57.) For $x = 0, 1, 2, \ldots$, we have

$$P[X = x] = \binom{n}{x}\left(\frac{1}{n}\right)^x\left(1 - \frac{1}{n}\right)^{n-x} \xrightarrow{n \to \infty} \frac{1}{x!e}.$$
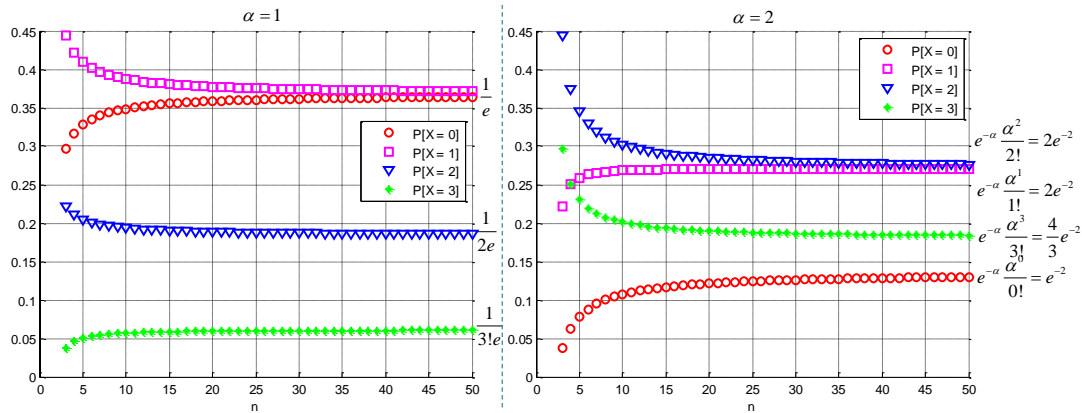

Figure 23: Pointwise convergence of the Binomial$\left(n, \frac{\alpha}{n}\right)$ pmf to the poisson pmf when $\alpha = 1$ and $\alpha = 2$.

Example 8.58. Consider $X \sim \mathcal{B}(n, \alpha/n)$. For $x = 0, 1, 2, \ldots$, we have

$$P[X = x] = \binom{n}{x}\left(\frac{\alpha}{n}\right)^x\left(1 - \frac{\alpha}{n}\right)^{n-x} \xrightarrow{n \to \infty} e^{-\alpha}\frac{\alpha^x}{x!}.$$

n=250

p = 1/1000

Example 8.59. In one of the New York state lottery games, a number is chosen at random between 0 and 999. Suppose you play this game 250 times. Use the Poisson approximation to estimate the probability that you will never win and compare this with the exact answer. [3, Q2.41]

Solution: Let W be the number of wins. Then, $W \sim \text{Binomial}(250, p)$ where $p = 1/1000$. Hence, $P[W = 0] = \binom{250}{0} p^0 (1-p)^{250} \approx 0.7787$.

If we approximate W by $\Lambda \sim \mathcal{P}(\alpha)$. Then we need to set $\alpha = np = \frac{250}{1000} = \frac{1}{4}$. In which case, $P[\Lambda = 0] = e^{-\alpha}\frac{\alpha^0}{0!} = e^{-\alpha} \approx 0.7788$ which is very close to the answer from direct calculation.

Example 8.60. Recall that Bortkiewicz applied the Poisson model to the number of Prussian cavalry deaths attributed to fatal horse kicks. Here, indeed, one encounters a very large number of trials (the Prussian cavalrymen), each with a very small probability of "success" (fatal horse kick).

8.61. Summary: *(minimal, default)*

$X \sim$	Support S_X	$p_X(x) = P[X = x]$
Uniform $\mathcal{U}(S)$	S	$\begin{cases} \frac{1}{\|S\|}, & x \in S, \\ 0, & \text{otherwise.} \end{cases}$
Bernoulli (p) $\quad \mathcal{B}(1,p)$	$\{0, 1\}$	$\begin{cases} 1-p, & x = 0, \\ p, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$
Binomial $\mathcal{B}(n, p)$	$\{0, 1, \ldots, n\}$	$\begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$
Geometric $\mathcal{G}_0(p)$	$\mathbb{N} \cup \{0\}$	$\begin{cases} p(1-p)^x, & x = 0, 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$
Geometric $\mathcal{G}_1(p)$	$\mathbb{N}$	$\begin{cases} p(1-p)^{x-1}, & x = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$
Poisson $\mathcal{P}(\alpha)$	$\mathbb{N} \cup \{0\}$	$\begin{cases} e^{-\alpha}\frac{\alpha^x}{x!}, & x = 0, 1, 2, \ldots, \\ 0, & \text{otherwise} \end{cases}$

family's name → (parameters) parameters

Table 3: Examples of probability mass functions. Here, $p \in (0, 1)$. $\alpha > 0$. $n \in \mathbb{N}$

8.4 Some Remarks

8.62. Sometimes, it is useful to define and think of pmf as a vector $\underline{\mathbf{p}}$ of probabilities.

When you use `MATLAB`, it is also useful to keep track of the values of x corresponding to the probabilities in $\underline{\mathbf{p}}$. This can be done via defining a corresponding vector $\underline{\mathbf{x}}$.

Example 8.63. For $\mathcal{B}\left(3, \frac{1}{3}\right)$, we may define

$$\underline{\mathbf{x}} = [0, 1, 2, 3]$$

and

$$\underline{\mathbf{p}} = \left[\binom{3}{0}\left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^3, \binom{3}{1}\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2, \binom{3}{2}\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1, \binom{3}{3}\left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^0 \right]$$

$$= \left[\frac{8}{27}, \frac{4}{9}, \frac{2}{9}, \frac{1}{27} \right]$$

8.64. At this point, we have a couple of ways to define probabilities that are associated with a random variable X

(a) We can define $P\left[X \in B\right]$ for all possible set B.

(b) For discrete random variable, we only need to define its pmf $p_X(x)$ which is defined as $P\left[X = x\right] = P\left[X \in \{x\}\right]$.

(c) We can also define the cdf $F_X(x)$.

Definition 8.65. If $p_X(c) = 1$, that is $P\left[X = c\right] = 1$, for some constant c, then X is called a **degenerated** random variable.